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Marshall - Olkin New Two-Parameter Sujatha Distribution and Its Applications.

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ABSTRACT
In this paper, we introduce a three parameter distribution called the Marshall-Olkin New Two-Parameter Sujatha Distribution. It is a Marshall - Olkin extension of the New two-parameter Sujatha distribution. Several important statistical properties including the hazard rate function, reversed hazard rate function and order statistics are derived. The application of this new distribution in the modeling of lifetime data is demonstrated using real-life data sets. A comparison of this new distribution with some of the existing distributions has also been shown.

KEYWORDS
Lifetime Distributions, Sujatha Distribution, Two Parameter Sujatha Distribution, Marshall - Olkin Transformation.

1. Introduction

The statistical modeling and analysis of lifetime data have gained increasing importance due to their widespread applications in fields such as engineering, insurance, and medical sciences. Over time, a variety of statistical methods, including both parametric and non-parametric approaches, have been developed to handle the challenges associated with lifetime data. Despite significant advancements, the field continues to evolve, with its scope and importance expanding further. Several probability distributions, such as Exponential, Gamma, Lognormal, Weibull, Lindley, Akash, Shanker, and Sujatha, are frequently used for modeling lifetime data. Each of these distributions offers distinct advantages and limitations when applied to real-world problems. For example, we cannot express the survival functions of the Lognormal and Gamma distributions in closed form. The Exponential distribution is characterized by a constant hazard rate. In comparison, the hazard rates for the Lindley, Shanker, Akash, and Sujatha distributions increase monotonically. These variations highlight the importance of selecting the appropriate distribution based on the specific characteristics of the data being analyzed.

Sujatha distribution(SD) has been introduced by [17]. It has been originally pre-

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sented as a one parameter continuous distribution with probability density function

$$f(x,\theta) = \frac{\theta^3}{\theta^2+\theta+2}(1+x+x^2)e^{-\theta x};\ x>0,\ \theta>0 \tag{1}$$

and cumulative distribution function

$$F(x,\theta) = 1 - \left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x};\ x>0,\ \theta>0. \tag{2}$$

In fact, this distribution is a three-component mixture of exponential (θ), gamma(2, θ) and gamma(3, θ) distributions. Generalizing this, [20] proposed a two-parameter Sujatha distribution (TPSD) having parameters θ and α which is defined by the probability density function

$$f(x,\theta,\alpha) = \frac{\theta^3}{\alpha\theta^2+\theta+2}(\alpha+x+x^2)e^{-\theta x};\ x>0,\ \theta>0,\ \alpha\geq 0 \tag{3}$$

and the cumulative distribution function

$$F(x,\theta,\alpha) = 1 - \left[1+\frac{\theta x(\theta x+\theta+2)}{\alpha\theta^2+\theta+2}\right]e^{-\theta x};\ x>0,\ \theta>0,\ \alpha\geq 0. \tag{4}$$

The New two-parameter Sujatha distribution(NTPSD) having pdf

$$f(x,\theta,\alpha) = \frac{\theta^3}{\theta^2+\alpha\theta+2}(1+\alpha x+x^2)e^{-\theta x};\ x>0,\ \theta>0,\ \alpha\geq 0 \tag{5}$$

and cdf

$$F(x,\theta,\alpha) = 1 - \left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x};\ x>0,\ \theta>0,\ \alpha\geq 0. \tag{6}$$

was also introduced by [21]

Here θ serves as the scale parameter while α irepresents the shape parameter. It is easy to verify that the two parameter-Sujatha distribution reduces to the Sujatha distribution for $\alpha = 1$.

The important properties of the Sujatha distribution, the new two-parameter Sujatha distribution and another two-parameter Sujatha distribution (ATPSD) including the hazard rate function, mean residual life function, stochastic ordering, Bonferroni and Lorenz curves, stress-strength reliability, have been thoroughly discussed by [**?**], [21] and [22]. The properties of a generalization of Sujatha distribution has been explored by [18].

This article presents the Marshall - Olkin generalization of the new two parameter Sujatha distribution. This type of generalization was originally proposed by [13] as a novel approach for extending existing families of distributions. This method consists of considering the survival function, say $\bar{F}(x)$ of an existing distribution and constructing a new survival function $\bar{G}(x)$ defined as

$$\bar{G}(x) = \frac{\bar{F}(x)}{\gamma + (1-\gamma)\bar{F}(x)}; \; 0 \leq \gamma \leq 1 \tag{7}$$

The function $\bar{G}(x)$ defines a new family of survival functions. This approach of extending an existing distribution often results in distributions with useful hazard functions. As a direct outcome, these extended distributions can be used to model real-life situations more effectively than previously known distributions. Due to their broad applicability, the Marshall-Olkin extended family of distributions has been extensively studied by numerous researchers such as [1], [2], [3], [5], [6], [12], [7], [11], [10], [15], [16] and [8]. The Marshall - Olkin Sujatha(MOS) distribution was introduced by [9]. Some of its useful statistical properties such as survival rate function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, stochastic ordering, Shannon and Renyi entropies, order statistics etc. were also derived. Recently, [4] introduced the Marshall-Olkin X Lindley distribution and demonstrated its superior adaptability to medical data compared to several existing probability models. More recently, [19] proposed the Marshall-Olkin two-parameter Sujatha distribution (MOTPSD) and found that it serves as a better model to some datasets compared to the existing SD, TPSD and ATPSD. This motivated us to study the Marshall-Olkin transformation of the NTPSD.

2. Marshall-Olkin New Two Parameter Sujatha Distribution

The survival function $\bar{G}(x,\ \theta,\ \alpha,\ \gamma)$ of the Marshall-Olkin New Two Parameter Sujatha Distribution (MONTPSD) computed using (7) is given by

$$\begin{aligned}\bar{G}(x,\ \theta,\ \alpha,\ \gamma) &= \frac{\bar{F}(x,\ \theta,\ \alpha)}{\gamma + (1-\gamma)\bar{F}(x,\ \theta,\ \alpha)} \\ &= \frac{\left[1 + \frac{\theta x(\theta x + \alpha\theta + 2)}{\theta^2 + \alpha\theta + 2}\right] e^{-\theta x}}{\gamma + (1-\gamma)\left[1 + \frac{\theta x(\theta x + \alpha\theta + 2)}{\theta^2 + \alpha\theta + 2}\right] e^{-\theta x}}.\end{aligned}$$

The cumulative distribution function of MONTPSD is given by

$$\begin{aligned}G(x,\ \theta,\ \alpha,\ \gamma) &= 1 - \bar{G}(x,\ \theta,\ \alpha) \\ &= \frac{\gamma\left\{1 - \left[1 + \frac{\theta x(\theta x + \alpha\theta + 2)}{\theta^2 + \alpha\theta + 2}\right] e^{-\theta x}\right\}}{\gamma + (1-\gamma)\left[1 + \frac{\theta x(\theta x + \alpha\theta + 2)}{\theta^2 + \alpha\theta + 2}\right] e^{-\theta x}}.\end{aligned} \tag{8}$$

Consequently, the probability density function of $MONTPSD$ $g(x,\ \theta,\ \alpha,\ \gamma)$ is

$$g(x,\ \theta,\ \alpha,\ \gamma) = \frac{\gamma\theta^3(1 + x^2 + \alpha x)e^{-\theta x}}{(\theta^2 + \alpha\theta + 2)\left\{\gamma + (1-\gamma)\left[1 + \frac{\theta x(\theta x + \alpha\theta + 2)}{\theta^2 + \alpha\theta + 2}\right] e^{-\theta x}\right\}^2}; \tag{9}$$

where $x > 0,\ \alpha \geq 0, \theta > 0,\ 0 < \gamma < 1$.

Figure 1 illustrates the plots of probability density function corresponding to different sets of parameter values.

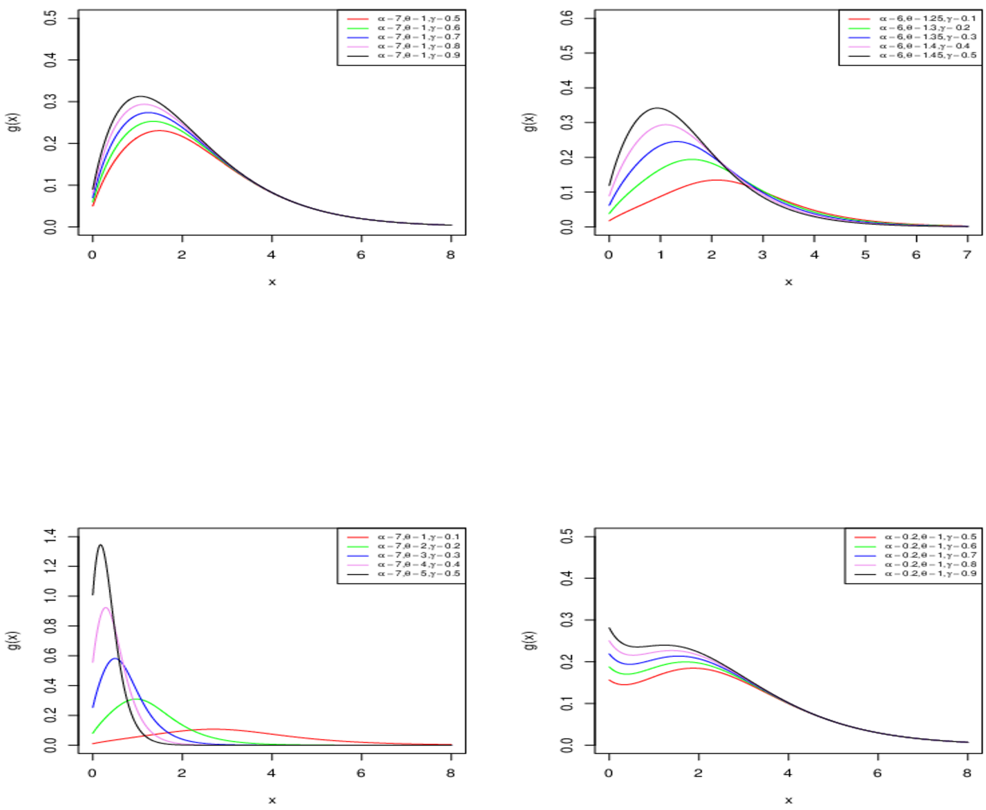


Figure 1. PDF of MONTPSD for varying parameters.

The hazard rate function is

$$
\begin{aligned}
h(x,\ \theta,\ \alpha,\ \gamma) &= \frac{g(x,\ \theta,\ \alpha,\ \gamma)}{\bar{G}(x,\ \theta,\ \alpha,\ \gamma)} \\
&= \frac{\gamma\theta^3(1+x^2+\alpha x)}{(\theta^2+\alpha\theta+2)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]} \\
&\times \frac{1}{\left\{\gamma+(1-\gamma)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}}
\end{aligned} \tag{10}
$$

The hazard rate function is depicted in figure 2. It is evident that the hazard rate is an increasing function of x.

The reverse hazard rate function is

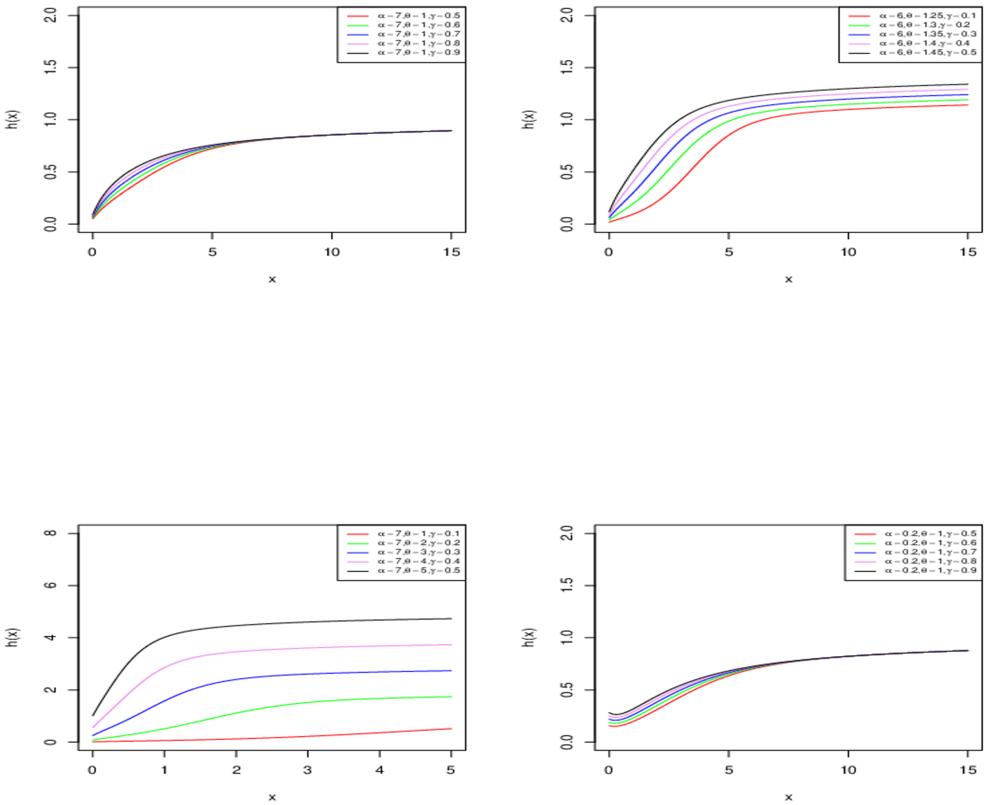


Figure 2. Hazard rate function of MONTPSD for varying parameters.

$$
\begin{aligned}
h_r(x,\theta,\alpha,\gamma) &= \frac{g(x,\theta,\alpha,\gamma)}{\bar{G}(x,\theta,\alpha,\gamma)} \\
&= \frac{\gamma\theta^3(1+x^2+\alpha x)}{(\theta^2+\alpha\theta+2)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]} \\
&\quad \times \frac{1}{\left\{\gamma+(1-\gamma)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}}
\end{aligned}
$$

3. Distribution of Order Statistics

Based on a random sample $X_1,\ X_2,\ \ldots,\ X_n$ of size n from the population having probability density function given by (9), the r^{th} order statistic $X_{(r)}$ has the probability density function

$$\begin{aligned} g_{x_{(r)}}(x,\ \theta,\ \alpha,\ \gamma) &= \frac{n!}{(r-1)!(n-r)!} g(x,\ \theta,\ \alpha,\ \gamma)[G(x,\ \theta,\ \alpha,\ \gamma)]^{r-1} \\ &\times \ [1-G(x,\ \theta,\ \alpha,\ \gamma)]^{n-r} \quad (11) \\ &= \frac{n!\gamma^r\theta^3(1+x^2+\alpha x)e^{-\theta x}\left\{\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}^{n-r}}{(r-1)!(n-r)!(\theta^2+\alpha\theta+2)} \\ &\times \frac{\left\{1-\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}^{r-1}}{\left\{\gamma+(1-\gamma)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}^{n+1}}; \quad (12) \\ & \quad x>0,\ \alpha\geq 0, \theta>0,\ 0<\gamma<1. \end{aligned}$$

Now, the first order statistic and the n^{th} order statistic $X_{(1)}$ and $X_{(n)}$ have the probability density functions:

$$\begin{aligned} g_{x_{(1)}}(x,\ \theta,\ \alpha,\ \gamma) &= \frac{n\gamma\theta^3(1+x^2+\alpha x)e^{-n\theta x}\left\{\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]\right\}^{n-1}}{(\theta^2+\alpha\theta+2)\left\{\gamma+(1-\gamma)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}^{n+1}}; \\ & \quad x>0,\ \alpha\geq 0, \theta>0,\ 0\leq\gamma\leq 1. \quad (13) \end{aligned}$$

and

$$\begin{aligned} g_{x_{(n)}}(x,\ \theta,\ \alpha,\ \gamma) &= \frac{n\gamma^n\theta^3(1+x^2+\alpha x)e^{-\theta x}\left\{1-\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]\right\}^{n-1}}{(\theta^2+\alpha\theta+2)\left\{\gamma+(1-\gamma)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}^{n+1}}; \\ & \quad x>0,\ \alpha\geq 0, \theta>0,\ 0\leq\gamma\leq 1. \quad (14) \end{aligned}$$

respectively.

4. Moments and related measures

We have

$$\begin{aligned} & \left\{\gamma+(1-\gamma)\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x}\right\}^{-2} \\ & \qquad = \gamma^{-2}\left\{1+\left(\frac{1-\gamma}{\gamma}\right)\right\} \\ & \times\left\{\left[\frac{(\theta x^2+\alpha\theta+2)+\theta x(\theta x+\alpha\theta+2)}{(\theta^2+\alpha\theta+2)}\right]e^{-\theta x}\right\}^{-2} \end{aligned}$$

$$
\begin{aligned}
&= \gamma^{-2}\sum_{k=0}^{\infty}(-1)^k\binom{k+1}{k}\left(\frac{1-\gamma}{\gamma}\right)^k\left[1+\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]^k e^{-\theta kx}\\
&= \gamma^{-2}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^k\left(\frac{1-\gamma}{\gamma}\right)^k\binom{k+1}{k}\binom{k}{i}\left[\frac{\theta x(\theta x+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]^i e^{-\theta kx}\\
&= \gamma^{-2}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^k\left(\frac{1-\gamma}{\gamma}\right)^k\binom{k+1}{k}\binom{k}{i}\left[\frac{\theta x}{\theta^2+\alpha\theta+2}\right]^i\\
&\times \sum_{j=0}^{i}\binom{i}{j}(\alpha\theta+2)^{i-j}(\theta x)^j e^{-\theta kx}\\
&= \gamma^{-2}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^k\left(\frac{1-\gamma}{\gamma}\right)^k\binom{k+1}{k}\binom{k}{i}\binom{i}{j}(\theta)^{i+j}(\theta^2+\alpha\theta+2)^{-i}\\
&\times (\alpha\theta+2)^{i-j}x^{i+j}e^{-\theta kx}
\end{aligned}
$$

Therefore

$$
g(x,\ \theta,\ \alpha\ ,\gamma) = \frac{\theta^3}{\gamma(\theta^2+\alpha\theta+2)}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}P_{kij}x^{i+j}e^{-\theta(k+1)x}(1+x^2+\alpha x); \qquad (15)
$$

$x > 0,\ \alpha \geq 0;\ \theta > 0,\ 0 < \gamma < 1$
where

$$
P_{kij} = (-1)^k\left(\frac{1-\gamma}{\gamma}\right)^k\binom{k+1}{k}\binom{k}{i}\binom{i}{j}(\theta)^{i+j}(\theta^2+\alpha\theta+2)^{-i}(\alpha\theta+2)^{i-j}
$$

Thus, the rth moment about the origin can be expressed as

$$
\begin{aligned}
\mu_r^{'} &= \frac{\theta^3}{\gamma(\theta^2+\alpha\theta+2)}\int_0^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}P_{kij}x^{i+j+r}e^{-\theta(k+1)x}(1+\alpha x+x^2)dx\\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}P_{kij}\frac{\theta^3}{\gamma(\theta^2+\alpha\theta+2)}\\
&\times \left\{\frac{\Gamma(i+j+r+1)}{[\theta(k+1)]^{i+j+r+1}}+\frac{\alpha\Gamma(i+j+r+2)}{[\theta(k+1)]^{i+j+r+2}}+\frac{\Gamma(i+j+r+3)}{[\theta(k+1)]^{i+j+r+3}}\right\} \qquad (16)
\end{aligned}
$$

In order to assess the distribution's characteristics with respect to dispersion, asymmetry and peakedness, the coefficient of variation (CV), the moment-based measure of skewness ($\sqrt{\beta_1}$) and the moment-based measure of kurtosis (β_2) have been computed for various parameter values as shown in table 1 below.

5. Estimation of Parameters

The parameters of Marshall - Olkin New Two parameter Sujatha distribution are estimated in this section by making use of the method of maximum likelihood. For

Table 1. Descriptive Statistics of Marshall-Olkin New Two-Parameter Sujatha Distribution

Parameters	μ_1'	μ_2'	μ_3'	μ_4'	CV	$\sqrt{\beta_1}$	β_2
$\alpha = 0.5$, $\theta = 1$, $\gamma = 0.5$	3.00797	12.6513	66.5043	413.159	0.63108	0.98997	4.17414
$\alpha = 2$, $\theta = 3$, $\gamma = 0.6$	0.66536	0.72420	1.05448	1.90399	0.79741	1.326	5.46722
$\alpha = 5$, $\theta = 0.5$, $\gamma = 0.6$	5.68603	45.0455	447.818	5323.84	0.62711	1.03883	4.58322
$\alpha = 7$, $\theta = 2$, $\gamma = 0.75$	1.06652	1.74559	3.7679	10.0477	0.73119	1.28427	5.42254

a random sample $X_1, X_2, \ldots, X_n$ from the population having probability density function given by (9), the likelihood function is given by

$$L(\underline{x};\ \theta,\ \alpha,\ \gamma) = \left[\frac{\gamma\theta^3}{(\theta^2+\alpha\theta+2)}\right]^n e^{-\theta\sum_{1=1}^{n} x_i}$$
$$\times \prod_{i=1}^{n}\left\{\frac{1+x_i^2+\alpha x_i}{\left\{\gamma+(1-\gamma)\left[1+\frac{\theta x_i(\theta x_i+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x_i}\right\}^2}\right\}$$

The log likelihood function is

$$\ln L(\underline{x};\ \theta,\ \alpha,\ \gamma) = n\ln(\gamma\theta^3) - n\ln(\theta^2+\alpha\theta+2) - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln(1+x_i^2+\alpha x_i)$$
$$-2\sum_{1=1}^{n}\ln\left\{\gamma+(1-\gamma)\left[1+\frac{\theta x_i(\theta x_i+\alpha\theta+2)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x_i}\right\} \tag{17}$$

The maximum likelihood estimation of the parameters α, θ and γ of the proposed MONTPSD using the dataset 1 given in section 7 was carried out with the NMaximize function in Mathematica. The procedure yielded stable and consistent estimates for the parameters. To examine the sensitivity of the log-likelihood function to initial values, the optimization was repeated with four distinct sets of starting points. As shown in table 2, the algorithm converged to the same MLEs and log-likelihood values in all cases, demonstrating the robustness and reliability of the estimation process with respect to initial parameter choices.

Table 2. Stability of Maximum Likelihood Estimates Under Varying Initial Guesses

Initial Guess	MLE	-Log Likelihood
$\alpha = 30, \theta = 0.20, \gamma = 0.05$	$\hat{\alpha} = 33.0636, \hat{\theta} = 0.1509, \hat{\gamma} = 0.0651$	206.415
$\alpha = 32, \theta = 0.15, \gamma = 0.06$	$\hat{\alpha} = 33.0636, \hat{\theta} = 0.1509, \hat{\gamma} = 0.0651$	206.415
$\alpha = 34, \theta = 0.25, \gamma = 0.07$	$\hat{\alpha} = 33.0636, \hat{\theta} = 0.1509, \hat{\gamma} = 0.0651$	206.415
$\alpha = 28, \theta = 0.18, \gamma = 0.04$	$\hat{\alpha} = 33.0636, \hat{\theta} = 0.1509, \hat{\gamma} = 0.0651$	206.415

6. Stress - Strength Reliability

Consider two independent random variables X and Y respectively representing the strength and stress of a component. If we assume that X and Y have the MONTPS distribution with parameters $(\theta_1, \alpha_1, \gamma_1)$ and $(\theta_2,\ \alpha_2,\ \gamma_2)$ respectively, then the stress - strength reliability can be obtained as

$$\begin{aligned} R = P(Y < X) &= \int_0^\infty g(x,\ \theta_1, \alpha_1, \gamma_1) G(x, \theta_2,\ \alpha_2,\ \gamma_2) dx \\ &= \int_0^\infty \left[\frac{\gamma_2 \left[1 - \left(1 + \frac{\theta_2 x(\theta_2 x + \alpha_2\theta_2 + 2)}{\theta_2^2 + \alpha_2\theta_2 + 2}\right) e^{-\theta_2 x}\right]}{\gamma_2 + (1-\gamma_2)\left[1 + \frac{\theta_2 x(\theta_2 x + \alpha_2\theta_2 + 2)}{\theta_2^2 + \alpha_2\theta_2 + 2}\right] e^{-\theta_2 x}} \right] \\ &\times \left[\frac{\gamma_1 \theta_1^3 (\alpha_1 + x^2 + \alpha_1 x) e^{-\theta_1 x}}{(\theta_1^2 + \alpha_1\theta_1 + 2)\left[\gamma_1 + (1-\gamma_1)\left[1 + \frac{\theta_1 x(\theta_1 x + \alpha_1\theta_1 + 2)}{\theta_1^2 + \alpha_1\theta_1 + 2}\right] e^{-\theta_1 x}\right]^2} \right] dx \end{aligned}$$

This integral has been computed using the NIntegrate function of Mathematica for some set of values of the parameters and the values of R so obtained is shown in table 3.

Table 3. Values of Stress - Strength Reliability(R)

Sl.No	θ_1	α_1	γ_1	θ_2	α_2	γ_2	**R**
1.	0.25	1	0.2	0.25	5	0.9	0.766436
2.	0.5	2	0.4	1	4	0.7	0.786854
3.	1	3	0.5	1.25	1	0.5	0.492820
4.	1.25	2	0.6	2	3	0.3	0.555466
5.	2	1	0.8	0.5	2	0.1	0.009283

7. Application

This section demonstrates the application of the Marshall-Olkin New Two-Parameter Sujatha distribution to real-life datasets. Dataset 1 represents the remission time (in months) of 50 breast cancer patients who were treated with trastuzumab, as documented by the Cancer Registry Department of the University of Benin Teaching Hospital located in Benin, Edo State, Nigeria. Dataset 2 represents the breaking stress of 65 carbon fibres of 50 mm length (GPa) reported by [14]. These datasets are modeled using MONTPSD and a comparison is subsequently conducted to show that it provides a better fit than the Sujatha distribution, the two-parameter Sujatha distribution, another two-parameter Sujatha distribution, and the new two-parameter Sujatha distribution.

Dataset 1
50, 74, 35, 39, 21, 37, 27, 35, 30, 35,
26, 28, 34, 34, 26, 41, 61, 33, 33, 26,
25, 41. 35, 34, 34, 33, 60, 61, 42, 30,
80, 31, 24, 49, 26, 31, 28, 41, 37, 41,
61, 33, 26, 34, 50, 73, 45, 80, 39, 21.

Dataset 2
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03,
2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59,
2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97,
3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33,
3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

In order to assess the goodness of fit of MONTPSD to these datasets, the parameters are first estimated by minimizing the negative log likelihood function. This is done by making use of the function NMinimize of Mathematica. In order to compare the performance of MONTPSD with Sujatha distribution (SD), Two-parameter Sujatha distribution (TPSD), Another two-parameter Sujatha distribution (ATPSD) and New two-parameter Sujatha distribution (NTPSD) we consider the criteria such as Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC) and Akaike Information Criterion Corrected (AICc). The distribution which shows lesser values of AIC, BIC, AICc is better. The values of these criteria are calulated using the following formulae:

$$AIC = 2k - 2\ln L,\ BIC = k\ln(n) - 2\ln L \text{ and } AICc = AIC + \frac{2k(k+1)}{n-k-1}$$

where n denotes the size of the sample and k stands for the number of parameters in the statistical model.
The results are summarized in table 4 and table 5.

Table 4. Model Comparison with Respect to Dataset 1

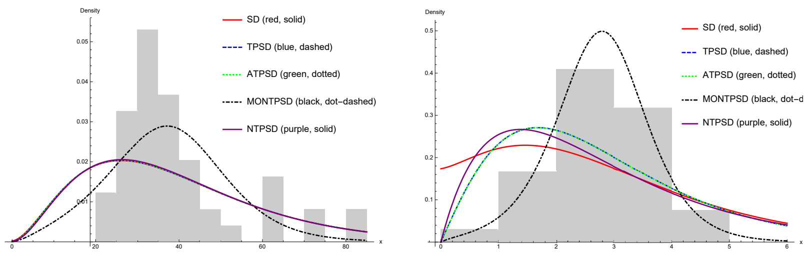
Distribution	MLE	AIC	BIC	AICc
SD	$\hat{\theta} = 0.0747$	422.482	424.394	422.565
TPSD	$\hat{\alpha} = 0.001$ $\hat{\theta} = 0.0748$	424.297	428.121	424.552
ATPSD	$\hat{\alpha} = 2.190$ $\hat{\theta} = 7.749$	424.297	428.122	424.553
NTPSD	$\hat{\alpha} = 10.920$ $\hat{\theta} = 0.0753$	429.715	433.539	429.971
MONTPSD	$\hat{\alpha} = 0.0013$ $\hat{\theta} = 0.157$ $\hat{\gamma} = 0.063$	414.419	420.155	414.941

Table 5. Model Comparison with Respect to Dataset 2

Distribution	MLE	AIC	BIC	AICc
SD	$\hat{\theta} = 0.853$	231.587	233.777	231.650
TPSD	$\hat{\alpha} = 0.001$ $\hat{\theta} = 0.969$	217.474	221.853	217.664
ATPSD	$\hat{\alpha} = 10.000$ $\hat{\theta} = 0.952$	219.571	223.95	219.761
NTPSD	$\hat{\alpha} = 0.000$ $\hat{\theta} = 0.969$	217.452	221.831	217.642
MONTPSD	$\hat{\alpha} = 10.000$ $\hat{\theta} = 3.363$ $\hat{\gamma} = 0.001$	-315.226	-308.657	-314.839

Figure 3 shows the plots of the probability density functions of these distributions overlayed on the histogram of the datasets 1 and 2.

Across both datasets, the MONTPSD consistently exhibited superior performance,



(a) Plots of fitted distributions for Dataset 1.

(b) Plots of fitted distributions for Dataset 2.

Figure 3. Plots of fitted distributions for Dataset 1 and Dataset 2.

as evidenced by the lowest values of AIC, BIC, and AICc, indicating enhanced model parsimony and goodness of fit. Complementary graphical analysis reinforced these quantitative results, with the MONTPSD's probability density function more closely aligning with the empirical data distribution and displaying minimal deviation, compared to the alternative models. These findings demonstrate the robustness and adaptability of the MONTPSD in effectively characterizing the underlying data structure, underscoring its suitability as a preferred model in analogous applications.

8. Simulation

To evaluate the performance of the maximum likelihood estimators (MLEs) for the MONTPSD, a simulation study was conducted using sample sizes n=20,50,100,500,1000, and 10,000. Three distinct sets of parameter values:

- $\alpha = 0.005,\ \theta = 0.99,\ \gamma = 0.45$
- $\alpha = 2,\ \theta = 0.5,\ \gamma = 0.75$
- $\alpha = 5,\ \theta = 3,\ \gamma = 0.25$

were considered to examine the behavior of the estimators under varying conditions. For each parameter configuration and sample size, simulated datasets were generated and the corresponding MLEs, along with their absolute bias and squared error were computed as shown in table 6. The consistency of MLEs are as evident as indicated by the reduction in the average absolute bias and mean square error with increase in sample size.

9. Summary and Discussion

A novel lifetime distribution, termed the Marshall-Olkin New Two-Parameter Sujatha (MONTPSD) distribution, is proposed and its statistical properties are comprehensively explored. Analytical expressions for the probability density function, cumulative distribution function, hazard rate function, reverse hazard rate function, and the distribution of order statistics are derived. Additionally, the behavior of skewness and kurtosis is examined to assess the distribution's flexibility in modeling data with varying degrees of asymmetry and peakedness. The parameters of the MONTPSD are estimated using the method of maximum likelihood, and the estimation procedure is outlined in detail. The practical utility of the proposed distribution is demonstrated through a comparative analysis with existing models, including the Sujatha distribution, the two-parameter Sujatha distribution, Another two-parameter Sujatha distribution, and the New two-parameter Sujatha distribution using two real-life datasets. Results demonstrate the superior performance of the MONTPSD in capturing the underlying structure of the observed data.

Table 6. Average Absolute Bias and Mean Squared Errors of MLEs of $MONTPSD(\alpha,\ \theta,\ ,\gamma)$.

Parameter		$n = 20$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$	$n = 10000$
$\alpha = 0.005$	Avg. Abs.Bias	4.99×10^{13}	1.214×10^{13}	1.310×10^{12}	0.374121	0.20079	0.04041
	MSE	2.573×10^{28}	3.614×10^{26}	2.076×10^{25}	0.48791	0.18030	0.00478
$\theta = 0.99$	Avg. Abs.Bias	0.22295	0.12887	0.10072	0.02679	0.02392	0.00654
	MSE	0.068374	0.02486	0.01490	0.00129	0.00102	7.03×10^{-5}
$\gamma = 0.45$	Avg. Abs.Bias	0.33717	0.17904	0.11353	0.040054	0.04145	0.01415
	MSE	0.26154	0.05165	0.019442	0.00275	0.00240	0.000325
$\alpha = 2$	Avg. Abs.Bias	3.74×10^{13}	2.96×10^{13}	1.24×10^{13}	2.92317	1.86726	0.53196
	MSE	2.04×10^{28}	7.11×10^{27}	1.36×10^{27}	18.63687	5.43549	0.42352
$\theta = 0.5$	Avg. Abs.Bias	0.16797	0.08798	0.05642	0.01526	0.01541	0.00454
	MSE	0.04523	0.01901	0.00410	0.00041	0.00037	3.5×10^{-5}
$\gamma = 0.75$	Avg. Abs.Bias	1.6683	0.72254	0.29909	0.13971	0.11392	0.04529
	MSE	5.55070	4.37393	0.12054	0.02601	0.01735	0.00428
$\alpha = 5$	Avg. Abs.Bias	7.7×10^{14}	4.43×10^{13}	2.06×10^{13}	4.69080	3.25830	0.60893
	MSE	7.99×10^{30}	6.12×10^{27}	3.04×10^{27}	58.61655	27.58962	0.53027
$\theta = 3$	Avg. Abs.Bias	0.85464	0.35233	0.31488	0.12587	0.09005	0.02779
	MSE	1.08364	0.18064	0.14363	0.02357	0.01438	0.00128
$\gamma = 0.25$	Avg. Abs.Bias	0.29038	0.15491	0.11894	0.04705	0.03425	0.01140
	MSE	0.18667	0.04222	0.01954	0.00347	0.00215	0.00025

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